JOURNAL OF APPROXIMATION THEORY 59, 87–106 (1989)

Properties of Minors of the Wronskian for Solutions of $L_n y + p(x) y = 0$ as Related to $(k, n-k)$ Disfocality

GARY D. JONES

Department of Mathematics and Statistics, Murray State University, Murray, Kentucky 42071, U.S.A.

AND

MARVIN KEENER

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74074, U.S.A.

Communicated by Oved Shisha

Received May 27, 1988

1. INTRODUCTION

We consider the differential equation

$$L_n y + p(x)y = 0, \tag{1.1}$$

where $p(x)$ is real valued, continuous, and of one sign on $[a, \infty)$. Also L_n is assumed to be a disconjugate linear differential operator. Thus L_n can be written as a product of first order linear operators. With [7] we let

$$L_0 y = \rho_0 y, \qquad L_i y = \rho_i (L_{i-1} y)', \qquad i = 1, ..., n, \tag{1.2}$$

with $\rho_i > 0$ and $\rho_i \in C^{n-i}$ for $i = 0, 1, ..., n$, and $\int^\infty \rho_i^{-1}(x)\, dx = \infty$ for $i = 1, ..., n-1$. We call $L_i y$ the ith quasi-derivative of y for $i = 0, 1, ..., n$.

Let $\{\alpha_0, \alpha_1, ..., \alpha_{n-1}\}$ and $\{\beta_0, \beta_1, ..., \beta_{n-1}\}$ be two sets of indices from $\{0, 1, ..., n-1\}$. If $I_k = \{\alpha_0, \alpha_1, ..., \alpha_{k-1}\}$ and $J_k = \{\beta_k, \beta_{k+1}, ..., \beta_{n-1}\}$, we consider boundary conditions on the interval $[a, b]$ of the form

$$L_i y(a) = 0, \qquad i \in I_k \tag{1.3}$$

$$L_i y(b) = 0, \qquad i \in J_k. \tag{1.4}$$

We will introduce some terminology and give some preliminary results.

0021-9045/89 \$3.00

Copyright © 1989 by Academic Press, Inc.
All rights of reproduction in any form reserved.

DEFINITION 1.1 [3]. Let $\sigma(c_0, ..., c_n)$ denote the number of sign changes in the sequence $c_0, ..., c_n$ of nonzero numbers. Then for a solution y of (1.1) that is not identically zero and a point x, we let

$$S(y, x^+) = \lim_{t \to x^+} \sigma(L_0 y(t), -L_1 y(t), ..., (-1)^n L_n y(t))$$

and

$$S(y, x^-) = \lim_{t \to x^-} \sigma(L_0 y(t), L_1 y(t), ..., L_n y(t)).$$

Let $a \leqslant x_1 \leqslant \cdots \leqslant x_r \leqslant b$ be the zeros of the quasi-derivatives $L_0 y, L_1 y, ..., L_{n-1} y$ of a nontrivial solution y of (1.1) in $[a, b]$, where the same $x_i = c$ is used to denote zeros of two different quasi-derivatives $L_j y$ and $L_k y$ if and only if $L_j y(c) = L_k y(c)$ implies either $L_s y(c) = 0$ for all $j \leqslant s \leqslant k$ or $L_s y(c) = 0$ for all $k \leqslant s \leqslant n-1$ and $0 \leqslant s \leqslant j$. With $n(x_i)$ denoting the number of consecutive (with $L_0 y$ following $L_{n-1} y$) quasi-derivatives which vanish at x_i, and $\langle q \rangle$ denoting the greatest even integer not greater than q, we state the following theorem.

THEOREM 1.1 [3]. *Every solution y of* (1.1) *satisfies the condition*

$$S(y, a^+) + \sum_{a < x_t < b} \langle n(x_t) \rangle + S(y, b^-) \leqslant n. \tag{1.5}$$

Moreover, $S(y, b^-)$ and $n - S(y, a^+)$ are both even if $p(x) < 0$ and both odd if $p(x) > 0$.

DEFINITION 1.2. The first extremal point $\theta_k(a)$ corresponding to the boundary conditions (1.3) and (1.4) is the first value of b in (a, ∞) for which there exists a nontrivial solution of (1.1), (1.3), and (1.4).

A necessary condition for the existence of $\theta_k(a)$ is that $n-k$ be even if $p(x) < 0$ and odd if $p(x) > 0$. In the following we will let γ be a positive integer less than n such that

$$(-1)^{n-\gamma} p(x) > 0. \tag{1.6}$$

It follows that $\theta_\gamma(a)$ fails to exist, while $\theta_{\gamma+1}(a)$ *may* exist.

When studying problems involving the existence of focal points (i.e., solutions of (1.3) and (1.4) where $\alpha_i = i = \beta_i$) a particular basis for the solution space of (1.1) is often constructed. While studying problems involving the existence of conjugate points (i.e., solutions of (1.3) and (1.4) where $\alpha_i = i$, $\beta_i = n-1-i$) a basis is often constructed in a different way.

In this paper we will show that certain elements of these bases are the same. In the process of showing that, we will study asymptotic properties of certain minors of the Wronskian of a basis for the solution space of (1.1). For related work, see [4, 5, 6].

Our first step will be to construct a two parameter family of bases for the solution space of (1.1). To that end let

$$\alpha_i = i, \qquad \beta_i = (i-j) \bmod (n) \qquad \text{for} \quad i = 0, 1, ..., n-1, \tag{1.7}$$

for j any fixed nonnegative integer less than or equal to γ. Thus

$$I_k = \{0, ..., k-1\} \qquad \text{and} \qquad J_k(j) = \{\beta_k, ..., \beta_{n-1}\}.$$

In this case, we will write $\theta_k(a, j)$ to emphasize the dependence of (1.4) on j.

Assuming (1.7), we define a basis

$$y_0(x, b), \qquad y_1(x, b), ..., \qquad y_{n-1}(x, b) \tag{1.8}$$

for the solution space of (1.1) as follows:

Let $y_{\gamma+2\eta}(x, b)$ be the essentially unique solution defined by

$$L_i y(a) = 0, \qquad i \in I_{\gamma+2\eta} \cup \{\gamma + 2\eta + 1\} \tag{1.9}$$

$$L_i y(b) = 0, \qquad i \in J_{\gamma+2\eta+2}(j). \tag{1.10}$$

Let $y_{\gamma+2\eta+1}(x, b)$ be defined by

$$L_i y(a) = 0, \qquad i \in I_{\gamma+2\eta+1} \tag{1.11}$$

$$L_i y(b) = 0, \qquad i \in J_{\gamma+2\eta+2}(j). \tag{1.12}$$

By letting b tend to infinity along a suitable sequence (see [6]), we can obtain another basis

$$y_0(x), \qquad y_1(x), ..., \qquad y_{n-1}(x) \tag{1.8$'$}$$

for the solution space of (1.1) from (1.8). When discussing this basis, we will use the notation of (1.8′) or that of (1.8) by saying that $b = \infty$.

The following theorem is a straightforward generalization of theorems that are in [6]. Thus the proof will be omitted. We will use the notation

$$W(y_{i_1}, y_{i_2}, ..., y_{i_k}; j) = \begin{vmatrix} L_j y_{i_1} & L_j y_{i_2} & \cdots & L_j y_{i_k} \\ L_{j+1} y_{i_1} & L_{j+1} y_{i_2} & \cdots & L_{j+1} y_{i_k} \\ \vdots & \vdots & \vdots & \vdots \\ L_{j+k-1} y_{i_1} & L_{j+k-1} y_{i_2} & \cdots & L_{j+k-1} y_{i_k} \end{vmatrix}.$$

THEOREM 1.2. *If γ satisfies (1.6), then the basis (1.8) or (1.8′) satisfies the following properties*:

1. $z \in \operatorname{span}\{y_\gamma(x, b), y_{\gamma+1}(x, b), \ldots, y_{\gamma+2s+1}(x, b)\}$ *for* $0 \leqslant s < (n-\gamma-1)/2$ *implies* $\gamma+1 \leqslant S(z, x^+) \leqslant (\gamma+2s+1)$ *and* $n-(\gamma+2s+1) \leqslant S(z, x^-) \leqslant n-(\gamma+1)$ *for* $x \in (a, b)$, *where* $b \leqslant \infty$.

1′. $z \in \operatorname{span}\{y_\gamma(x, b), y_{\gamma+1}(x, b), \ldots, y_{n-1}(x, b)\}$ *implies* $\gamma+1 \leqslant S(z, x^+)$ *and* $S(z, x^-) \leqslant n-(\gamma+1)$ *for* $x \in (a, b)$, *where* $b \leqslant \infty$.

2. $W(y_\gamma(x, b), y_{\gamma+1}(x, b), \ldots, y_{\gamma+2s+1}(x, b); i) \neq 0$ *for* $x \in (a, b)$, *where* $0 \leqslant i \leqslant n-2s-2$.

Statement (2) *is valid replacing* $\gamma+2s+1$ *with* $n-1$, *where* $0 \leqslant i \leqslant \gamma$.

2. Zeros of Minors and Existence of Extreme Points

In this section we will consider the basis (1.8) for the solution space of (1.1). We will assume (1.6) and (1.7) throughout.

Our purpose will be to show that the existence of $\theta_{\gamma+1}(a, j)$ is implied by the vanishing of certain minors of the Wronskian of (1.8). These results generalize those found in [6].

Let

$$W(x, s) = \begin{vmatrix} L_{\gamma-j+1}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+1}\, y_{\gamma+2k+1}(s, b) \\ \vdots & \vdots & \vdots \\ L_{\gamma-j+2k+1}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+2k+1}\, y_{\gamma+2k+1}(s, b) \\ L_\gamma\, y_\gamma(x, b) & \cdots & L_\gamma\, y_{\gamma+2k+1}(x, b) \end{vmatrix}, \tag{2.1}$$

$$u_1(x, s) = \begin{vmatrix} L_{\gamma-j+1}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+1}\, y_{\gamma+2k+1}(s, b) \\ \vdots & \vdots & \vdots \\ L_{\gamma-j+2k}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+2k}\, y_{\gamma+2k+1}(s, b) \\ y_\gamma(x, b) & \cdots & y_{\gamma+2k+1}(x, b) \\ L_\gamma\, y_\gamma(a, b) & \cdots & L_\gamma\, y_{\gamma+2k+1}(a, b) \end{vmatrix}, \tag{2.2}$$

$$u_2(x, s) = \begin{vmatrix} L_{\gamma-j+1}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+1}\, y_{\gamma+2k+1}(s, b) \\ \vdots & \vdots & \vdots \\ L_{\gamma-j+2k+1}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+2k+1}\, y_{\gamma+2k+1}(s, b) \\ y_\gamma(x, b) & \cdots & y_{\gamma+2k+1}(x, b) \end{vmatrix}, \tag{2.3}$$

and

$$u_3(x, s) = \begin{vmatrix} L_{\gamma-j+1}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+1}\, y_{\gamma+2k+1}(s, b) \\ \vdots & \vdots & \vdots \\ L_{\gamma-j+2k}\, y_\gamma(s, b) & \cdots & L_{\gamma-j+2k}\, y_{\gamma+2k+1}(s, b) \\ y_\gamma(x, b) & \cdots & y_{\gamma+2k+1}(x, b) \\ L_{\gamma+1}\, y_\gamma(a, b) & \cdots & L_{\gamma+1}\, y_{\gamma+2k+1}(a, b) \end{vmatrix}, \tag{2.4}$$

THEOREM 2.1. *Suppose for some* $s_0 \in (a, b)$, $W(a, s_0) = 0$. *Then the function* $x(s)$ *defined by* $W(x, s) = 0$ *where* $x(s_0) = a$ *is such that*

$$\left.\frac{dx}{ds}\right|_{s = s_0} > 0.$$

Proof. $\rho_{\gamma+1}(a)\,(\partial W/\partial x)|_{(a,s_0)} = L_{\gamma+1}u_2(a, s_0)$. Hence we need to show $L_{\gamma+1}u_2(a, s_0) \neq 0$. Suppose $L_{\gamma+1}u_2(a, s_0) = 0$. Then there is a solution v_1 in $\operatorname{span}\{y_\gamma, y_{\gamma+1}, ..., y_{\gamma+2k+1}\}$ with $L_i v_1(a) = 0$ for $i = 0, ..., \gamma-1, \gamma+1$; $L_i v_1(s_0) = 0$ for $i = \gamma - j + 1, ..., \gamma - j + 2k + 1$; and $L_i v_1(b) = 0$ for $i = \gamma - j + 2k + 2, ..., n - 1 - j$. Since $W(a, s_0) = 0$, there is a solution v_2 in $\operatorname{span}\{y_\gamma, y_{\gamma+1}, ..., y_{\gamma+2k+1}\}$ with $L_i v_2(a) = 0$ for $i = 0, ..., \gamma$; $L_i v_2(s_0) = 0$ for $i = \gamma - j + 1, ..., \gamma - j + 2k + 1$; and $L_i v_2(b) = 0$ for $i = \gamma - j + 2k + 2, ..., n - j - 1$. If $v_1 = v_2$, then $n \geqslant S(v_1, a^+) + \langle n(s_0)\rangle + S(v_1, b^-) \geqslant (\gamma + 3) + 2k + (n - 2k - 2 - \gamma) = n + 1$, which is not possible. If $v_1 \neq v_2$, then there is a linear combination z of v_1 and v_2 such that $L_{\gamma - j + 2k + 2}z(s_0) = 0$. Then $n \geqslant S(z, a^+) + \langle n(s_0)\rangle + S(z, b^-) \geqslant (\gamma + 1) + (2k + 2) + (n - 2k - 2 - \gamma) = n + 1$. Since that is not possible, it follows that

$$L_{\gamma+1}u_2(a, s_0) \neq 0. \tag{2.5}$$

Since $W(a, s_0) = 0$, it follows that for $m = 1$ or 2 that $L_i u_m(a, s_0) = 0$ for $i = 0, ..., \gamma - 1$; $L_i u_m(s_0, s_0) = 0$ for $i = \gamma - j + 1, ..., \gamma - j + 2k + 1$; and $L_i u_m(b, s_0) = 0$ for $i = \gamma - j + 2k + 2, ..., n - j - 1$. Since solutions satisfying such boundary conditions are essentially unique,

$$u_1(x, s_0) = c u_2(x, s_0). \tag{2.6}$$

Now the Wronskian

$$w(L_{i_0}u_2(x_1, s_0), L_{i_0}u_3(x_1, s_0)) \neq 0 \qquad \text{for} \quad x_1 \in (a, s_0). \tag{2.7}$$

Otherwise, there is a solution z in $\operatorname{span}\{u_2, u_3\}$ such that $L_i z(a, s_0) = 0$ for $i = 0, ..., \gamma - 1$; $L_i z(x_1, s_0) = 0$ for $i = i_0, i_0 + 1$; $L_i z(s_0, s_0) = 0$ for $i = \gamma - j + 1, ..., \gamma - j + 2k$; and $L_i z(b, s_0) = 0$ for $i = \gamma - j + 2k + 2, ..., n - j - 1$. In that case $n \geqslant S(z, a^+) + \langle n(x_1)\rangle + \langle n(s_0)\rangle + S(z, b^-) \geqslant (\gamma + 1) + 2 + 2k + (n - 2k - 2 - \gamma) = n + 1$, which is not possible.

By (2.7) and (2.6) the zeros of $L_{\gamma+1}u_1(x, s_0)$ and $L_{\gamma+1}u_3(x, s_0)$ separate on (a, s_0). Now $L_{\gamma+1}u_1(a, s_0) \neq 0$ by (2.5) and (2.6), but $L_{\gamma+1}u_3(a, s_0) = 0$. Since

$$L_\gamma u_3(a, s_0) = -L_{\gamma+1}u_1(a, s_0) \tag{2.8}$$

it follows that $L_\gamma u_3(a, s_0) \neq 0$.

By (1.7), $j \leqslant \gamma$. If $j > 0$, we wish to show that if $L_{\gamma - j + 1}u_1(x, s_0)$

has t zeros on (a, s_0), then $L_{\gamma-j+1}u_3(x, s_0)$ has $t+1$ zeros there. By (2.7) it is enough to show that if s_1 and s_2 are two consecutive zeros of $L_{\gamma-j+1}u_1(x, s_0)$ with either $s_1 = a$ or $s_2 = s_0$ on $[a, s_0]$, then $L_{\gamma-j+1}u_3(x, s_0)$ has a zero on (s_1, s_2). Suppose $L_{\gamma-j+1}u_3(x, s_0) \neq 0$ for $x \in (s_1, s_2)$. Then $h(x) \equiv L_{\gamma-j+1}u_1(x, s_0)/L_{\gamma-j+1}u_3(x, s_0)$ is continuous on (s_1, s_2). Now $L_{\gamma-j+1}u_1(x, s_0)$ has a zero of order j at $x = a$ while $L_{\gamma-j+1}u_3(x, s_0)$ has a zero of order $j-1 \geqslant 0$. At $x = s_0$, $L_{\gamma-j+1}u_1(x, s_0)$ and $L_{\gamma-j+1}u_3(x, s_0)$ have zeros of order $2k+1$ and $2k$, respectively. Thus, defining $h(s_1) = h(s_2) = 0$, we see by l'Hospital's rule that $h(x)$ is continuous on $[s_1, s_2]$. Since $h(x) \neq 0$ for $x \in (s_1, s_2)$, h must have an extreme point at s^* in (s_1, s_2) at which $h'(s^*) = 0$. It follows that

$$L_{\gamma-j+1}u_3(s^*, s_0)\, L_{\gamma-j+2}u_1(s^*, s_0) - L_{\gamma-j+1}u_1(s^*, s_0)\, L_{\gamma-j+2}u_3(s^*, s_0) = 0.$$

Thus $z(x) \equiv u_3(x, s_0)\, h(s^*) - u_1(x, s_0)$ is such that $L_{\gamma-j+1}z(s^*) = L_{\gamma-j+2}z(s^*) = 0$, which is not possible by (2.7). Thus $L_{\gamma-j+1}u_3(x, s_0)$ has a zero in (s_1, s_2).

By (2.7) and (2.8) $L_\gamma u_3(a, s_0) \neq 0$. Thus assume without loss of generality, that $L_\gamma u_3(a, s_0) > 0$. Since $L_i u_3(a, s_0) = 0$ for $i = 0, 1, ..., \gamma-1$, it follows that $L_i u_3(a^+, s_0) > 0$ for $i = 0, 1, ..., \gamma$. Since by (2.8) $L_\gamma u_3(a, s_0) = -L_{\gamma+1}u_1(a, s_0)$, it follows that $L_i u_1(a^+, s_0) < 0$ for $i = 0, 1, ..., \gamma+1$. If $L_{\gamma-j+1}u_1(x, s_0)$ has t zeros, which are necessarily simple in (a, s_0), then $(-1)^{t+1} L_{\gamma-j+1}u_1(s_0^-, s_0) > 0$. Because $L_i u_1(s_0, s_0) = 0$ for $i = \gamma-j+1, ..., \gamma-j+2k+1$, it follows that

$$(-1)^{t+1+i} L_{\gamma-j+1+i}u_1(s_0^-, s_0) > 0 \qquad \text{for} \quad i = 0, 1, ..., 2k+1. \quad (2.9)$$

Now, $L_{\gamma-j+1}u_3(x, s_0)$ has $t+1$ simple zeros in (a, s_0) implies $(-1)^{t+1} L_{\gamma-j+1}u_3(s_0^-, s_0) > 0$. Also, $L_i u_3(s_0, s_0) = 0$ for $i = \gamma-j+1, ..., \gamma-j+2k$ implies

$$(-1)^{t+1+i} L_{\gamma-j+1+i}u_3(s_0^-, s_0) > 0 \qquad \text{for} \quad i = 0, 1, ..., 2k. \quad (2.10)$$

Since $L_{\gamma+1}u_2(a, s_0) = L_{\gamma-j+2k+1}u_3(s_0, s_0)$, it follows that

$$\frac{\partial W(a, s_0)}{\partial s} = \frac{L_{\gamma-j+2k+2}u_1(s_0, s_0)}{\rho_{\gamma-j+2k+2}(s_0)}$$

and

$$\frac{\partial W(a, s_0)}{\partial x} = \frac{L_{\gamma+1}u_2(a, s_0)}{\rho_{\gamma+1}(a)} = \frac{L_{\gamma-j+2k+1}u_3(s_0, s_0)}{\rho_{\gamma+1}(a)}.$$

Thus

$$\frac{\partial W(a, s_0)/\partial s}{\partial W(a, s_0)/\partial x} = \frac{(-1)^{t+2k+2}\,(L_{\gamma-j+2k+2}u_1(s_0, s_0)/\rho_{\gamma-j+2k+2}(s_0))}{(-1)(-1)^{t+2k+1}\,(L_{\gamma-j+2k+1}u_3(s_0, s_0)/\rho_{\gamma+1}(a))} < 0.$$

Thus

$$\left.\frac{dx}{ds}\right|_{s=s_0} > 0.$$

If $j=0$, then since the zeros of $L_{\gamma+1}u_1(x, s_0)$ and $L_{\gamma+1}u_3(x, s_0)$ separate on (a, s_0), $L_{\gamma+1}u_1(a, s_0) \neq 0$, $L_{\gamma+1}u_3(a, s_0)=0$, and $L_\gamma u_3(a, s_0) \neq 0$, it follows that the first zero of $L_{\gamma+1}u_1(x, s_0)$ must precede the first zero of $L_{\gamma+1}u_3(x, s_0)$ on (a, s_0). Now $L_i u_1(s_0, s_0)=0$ for $i=\gamma+1, ..., \gamma+2k+1$. Also $L_{\gamma+2k+2}u_1(s_0, s_0) \neq 0$, otherwise $n \geqslant S(u_1, a^+) + \langle n(s_0)\rangle + S(u_1, b^-) \geqslant (\gamma+1)+(2k+2)+(n-2k-2-\gamma)=n+1$. Further, $L_i u_3(s_0, s_0)=0$ for $i=\gamma+1, ..., \gamma+2k$ and $L_{\gamma+2k+1}u_3(s_0, s_0) \neq 0$. Otherwise either u_2 and u_3 are linearly dependent or there is a solution v in span$\{u_2, u_3\}$ such that $L_{\gamma+2k+2}v(s_0, s_0)=0$. Hence $n \geqslant S(v, a^+) + \langle n(s_0)\rangle + S(v, b^-) \geqslant (\gamma+1)+(2k+2)+(n-2k-2-\gamma)=n+1$. If u_2 and u_3 are linearly dependent then $n \geqslant S(u_3, a^+) + \langle n(s_0)\rangle + S(u_3, b^-) \geqslant (\gamma+3)+2k+(n-2k-2-\gamma)=n+1$.

Since the order of the zero of $L_{\gamma+1}u_3(x, s_0)$ is less than the order of the zero of $L_{\gamma+1}u_1(x, s_0)$ at $x=s_0$, the last zero of $L_{\gamma+1}u_1(x, s_0)$ is before the last zero of $L_{\gamma+1}u_3(x, s_0)$ in (a, s_0). Thus if $L_{\gamma+1}u_1(x, s_0)$ has t zeros on (a, s_0) then so does $L_{\gamma+1}u_3(x, s_0)$. Now

$$-\left.\frac{dx}{ds}\right|_{s=s_0} = \frac{\partial W/\partial s}{\partial W/\partial x} = \frac{L_{\gamma+2k+2}u_1(s_0, s_0)/\rho_{\gamma+2k+2}(s_0)}{L_{\gamma+1}u_2(a, s_0)/\rho_{\gamma+1}(a)}$$

Suppose, without loss of generality, that $L_{\gamma+1}u_1(a, s_0)>0$. Then $L_{\gamma+1}u_1(s_0^-, s_0)(-1)^t>0$. Since $L_\gamma u_3(a, s_0) \neq 0$, then $n \geqslant S(u_3, a^+) + \langle n(s_0)\rangle + S(u_3, b^-) \geqslant (\gamma+1)+2k+(n-2k-2-\gamma)=n-1$. Hence $S(u_3, a^+)=\gamma+1$. Since $L_{\gamma+1}u_1(a, s_0)>0$, by (2.8) $L_\gamma u_3(a, s_0)<0$ from which it follows that $L_{\gamma+1}u_3(a^+, s_0)>0$. Hence, $L_{\gamma+1}u_3(s_0^-, s_0)(-1)^t>0$. Now $L_{\gamma+1}u_1(s_0^-, s_0)(-1)^t>0$ implies $L_{\gamma+2k+2}u_1(s_0, s_0)(-1)^{t+1}>0$. Note that $L_{\gamma+1}u_2(a, s_0)=L_{\gamma+2k+1}u_3(s_0, s_0)$. And $L_{\gamma+1}u_3(s_0^-, s_0)(-1)^t>0$ implies $L_{\gamma+2k+1}u_3(s_0, s_0)(-1)^t>0$. Hence

$$\begin{aligned}-\left.\frac{dx}{ds}\right|_{s=s_0} &= \frac{L_{\gamma+2k+2}u_1(s_0, s_0)/\rho_{\gamma+2k+2}(s_0)}{L_{\gamma+1}u_2(a, s_0)/\rho_{\gamma+1}(a)} \\ &= \frac{L_{\gamma+2k+2}u_1(s_0, s_0)/\rho_{\gamma+2k+2}(s_0)}{L_{\gamma+2k+1}u_3(s_0, s_0)/\rho_{\gamma+1}(a)} < 0.\end{aligned}$$

Let

$$D(x, s) \equiv W(y_{\gamma+1}(x, s), y_{\gamma+2}(x, s), ..., y_{\gamma+2k+1}(x, s); \gamma-j+1).$$

THEOREM 2.2. *There is* $s_0 \in (a, b)$ *such that* $D(s_0, b)=0$ *if and only if* $W(a, s_0)=0$.

Proof. Since y_i has a zero of order i at $x=a$, expanding $W(a, s_0)$ by the last row, we have $W(a, s_0) = -L_\gamma y_\gamma(a, b)\, D(s_0, b)$.

THEOREM 2.3. *Let $s \in (a, b)$ and $u_2(x, s)$ be given by* (2.3). *Then*

(i) *the simple zeros of $L_i u_2(x, s)$ are differentiable functions of s, and*

(ii) *a zero of $L_\gamma u_2(x, s)$ enters (a, s) through a at $s = s_0$ if $D(s_0, b) = 0$. No zero of $L_\gamma u_2(x, s)$ leaves the interval (a, s).*

Proof. Part (i) follows directly from the definition of $u_2(x, s)$ and the implicit function theorem.

By Theorem 2.1 a zero x_0 of $L_\gamma u_2(x, s)$ enters (a, s) as s increases through s_0. Again by Theorem 2.1, x_0 cannot exit (a, s) through a as s increases.

If $j=0$, then $L_\gamma u_2(s^*, s^*) \neq 0$ for $a < s^* < b$. Otherwise, $n \geqslant S(u_2, a^+) + \langle n(s^*) \rangle + S(u_2, b^-) \geqslant (\gamma+1) + (2k+2) + [n - (\gamma + 2k + 2)] = n+1$. Thus in case $j=0$, x_0 cannot exit (a, s) through s as s increases from s_0 to b.

If $j>0$ and $\gamma \leqslant \gamma - j + 2k + 1$, then by Rolle's theorem there exist zeros $x_0 < x_1 < \cdots < x_{2k+2-j}$ of $L_\gamma u_2(x, s), \ldots, L_{\gamma+2k+2-j} u_2(x, s)$ in (a, s). Since the zeros of $L_\gamma u_2(x, s)$ are simple, it follows that if $L_\gamma u_2(s^*, s^*) = 0$ then there is $s_1 \leqslant s^*$ so that $L_{\gamma+2k+2-j} u_2(s_1, s_1) = 0$. Hence $n \geqslant S(u_2, a^+) + \langle n(s_1) \rangle + S(u_2, b^-) \geqslant (\gamma+1) + (2k+2) + [n - (\gamma+2k+2)] = n+1$.

If $j>0$ and $\gamma > \gamma - j + 2k + 1$, then again applying Rolle's theorem there exist zeros $x_0 < x_{-1} < \cdots < x_{2k+2-j}$ of $L_\gamma u_2(x, s)$, $L_{\gamma-1} u_2(x, s), \ldots, L_{\gamma+2k+2-j} u_2(x, s)$ in (a, s). Since the zeros of $L_\gamma u_2(x, s)$ are simple, it follows that if $L_\gamma u_2(s^*, s^*) = 0$ then there is $s_2 \leqslant s^*$ so that $L_{\gamma+2k+2-j} u_2(s_2, s_2) = 0$. Hence $n \geqslant S(u_2, a^+) + \langle n(s_2) \rangle + S(u_2, b^-) \geqslant (\gamma+1) + (2k+2) + [n - (\gamma+2k+2)] = n+1$.

THEOREM 2.4. *Suppose there is an $s_0 \in (a, b)$ such that $D(s_0, b) = 0$. Then $\theta_{\gamma+1}(a, j)$ exists on $[a, b]$.*

Proof. First suppose $b < \infty$. Let $y(x, s) = \alpha(s)\, u_2(x, s)$ be such that $\sum_{i=0}^{n-1} (L_i y(a, s))^2 = 1$. Then by standard compactness arguments there is a sequence $\{s_i\}$ such that $\lim_{i\to\infty} s_i = b$ and $\lim_{i\to\infty} y(x, s_i) = z(x)$ is a nontrivial solution of (1.1) with convergence uniform on $[a, b]$. It follows that

$$L_i z(a) = 0 \qquad \text{for} \quad i = 0, 1, \ldots, \gamma - 1 \tag{2.11}$$

and

$$L_i z(b) = 0 \qquad \text{for} \quad i = \gamma - j + 1, \ldots, n - 1 - j. \tag{2.12}$$

By Theorem 2.3, a zero of $L_\gamma u_2(x, s)$ and thus $L_\gamma y(x, s)$ enters (a, s) through a at $s = s_0$. There is a subsequence of $\{s_i\}$, say $\{s_{i_k}\}$, such that

$\lim_{k \to \infty} x_m(s_{i_k}) = x_m^* \in [a, b]$, where x_m is the mth zero of $L_\gamma u_2(x, s)$ in (a, s). Because of (2.11), (2.12), and (1.5) each x_m^* must be distinct. It follows that $L_\gamma z(x_m^*) = 0$. Now $x_m^* \neq b$ for all m, otherwise if $j = 0$, $\theta_\gamma(a, j)$ exists, which is not possible. If $j > 0$, applying Rolle's theorem as in Theorem 2.3, we see that $\theta_\gamma(a, j-1)$ exists which is also impossible. If $x_m^* = a$ for some m, then $\theta_{\gamma+1}(a, j)$ exists and we are through. If $x_m^* \in (a, b)$ for all m, define

$$z(x, s) \equiv \begin{vmatrix} L_0 z_1(a) & \cdots & L_0 z_n(a) \\ \vdots & \vdots & \vdots \\ L_{\gamma-1} z_1(a) & \cdots & L_{\gamma-1} z_n(a) \\ L_{\gamma-j+1} z_1(s) & \cdots & L_{\gamma-j+1} z_n(s) \\ \vdots & \vdots & \vdots \\ L_{n-1-j} z_1(s) & \cdots & L_{n-1-j} z_n(s) \\ z_1(x) & \cdots & z_n(x) \end{vmatrix},$$

where $z_1, \dots, z_n$ is a basis for the solution space of (1.1). Since z satisfies (2.11) and (2.12) and such solutions are essentially unique, it follows that $z(x, b) = kz(x)$. It follows easily that the zeros of $L_i z(x, s)$ in (a, s) are simple and thus differentiable functions of s. Let $x_m^*(s)$ be the simple zero of $L_\gamma z(x, s)$ so that $x_m^*(b) = x_m^*$. Normalizing $z(x, s)$ so that is does not converge to the trivial solution as s tends to a, it follows that as s decreases toward a, one of these zeros must exit (a, s). As above, it cannot exit through s, thus it must exit through a and hence $\theta_{\gamma+1}(a, j)$ exists in $[a, b]$.

For the infinite case, we note that the basis (1.8′) satisfying the conditions of Theorem 1.2 is of the form

$$y_i(x) = \lim_{m \to \infty} y_i(x, b_m),$$

where $\{b_m\}$ diverges to infinity. Further, $y_i(x, b_m)$ and its quasi-derivatives converge uniformly to $y_i(x)$ and its quasi-derivatives on compact intervals. Hence

$$\lim_{m \to \infty} D(x, b_m) = D(x) \equiv W(y_{\gamma+1}(x), y_{\gamma+2}(x), \dots, y_{\gamma+2k+1}(x); \gamma - j + 1).$$

If $D(s_0) = 0$, then $D'(s_0) \neq 0$. Otherwise, there is a solution $u \in \{y_{\gamma+1}(x), y_{\gamma+2}(x), \dots, y_{\gamma+2k+1}(x)\}$ with $\langle n(s_0) \rangle = 2k+2$, which is not possible by Theorems 1.1 and 1.2. Since $D(x)$ changes signs at s_0, there is an m such that $D(x, b_m) = 0$ for some $x \in (a, b_m)$. Thus from the finite case $\theta_{\gamma+1}(a, j)$ exists on $[a, b_m)$ and thus on $[a, +\infty)$.

3. Asymptotic Properties of Minors and Bases for the Solution Space of (1.1)

In this section we will show that certain elements of the bases (1.8′) are independent of the boundary conditions at b.

For a fixed j we will let

$$y_0(x, b), \qquad y_1(x, b), ..., \qquad y_{n-1}(x, b)$$

be the basis (1.8).

Choose b close enough to a so that no nontrivial solution of (1.1) has more than $n-1$ quasi-derivatives that vanish on $[a, b]$. We replace the basis (1.8) with

$$u_0(x, b), \qquad u_1(x, b), ..., \qquad u_{n-1}(x, b) \tag{3.1}$$

by letting

$$u_i(x, b) = y_i(x, b) \qquad \text{if } i-\gamma \text{ is odd}$$

$$u_i(x, b) = y_i(x, b) - \frac{L_{i+1-j}\, y_i(b, b)}{L_{i+1-j}\, y_{i+1}(b, b)}\, y_{i+1}(x, b) \qquad \text{if } i-\gamma \text{ is even.}$$

Since (1.1) is assumed to have no extreme point on $[a, b]$, $L_{i+1-j}\, y_{i+1}(b, b) \neq 0$. It easily follows that (3.1) is a basis for the solution space of (1.1).

Theorem 3.1. *Let $b > a$ be as above. The set $\{L_{\gamma-j}u_\gamma(x, b), L_{\gamma-j}u_{\gamma+1}(x, b), ..., L_{\gamma-j}u_{n-1}(x, b)\}$ (replacing $u_i(x, b)$ by $-u_i(x, b)$ if necessary) is a Markov system on $[a, b)$.*

Proof. Suppose $W(u_\gamma(c, b), u_{\gamma+1}(c, b), ..., u_{\gamma+k}(c, b); \gamma-j) = 0$ for $c \in [a, b)$. Then there is $y \in \operatorname{span}\{u_\gamma(x, b), u_{\gamma+1}(x, b), ..., u_{\gamma+k}(x, b)\}$ so that

$$L_i\, y(a, b) = 0, \qquad i = 0, 1, ..., \gamma-1$$

$$L_i\, y(c, b) = 0, \qquad i = \gamma-j, \gamma-j+1, ..., \gamma-j+k$$

$$L_i\, y(b, b) = 0, \qquad i = \gamma+k+1-j, \gamma+k+2-j, ..., n-1-j.$$

Hence (1.1) has a solution with $\gamma + (k+1) + (n-\gamma-k-1) = n$ vanishing quasi-derivatives on $[a, b]$, contrary to the hypothesis.

We next state a sequence of lemmas that are generalizations of those found in [1]. Since the proofs are essentially the same as in [1] they will be omitted.

In these lemmas we will let z and

$$\{z_1, z_2, ..., z_n\}$$

be admissible functions.

LEMMA 3.1. *If* $W(z_1, z_2, ..., z_{k-1}; i) \neq 0$ *and* $W(z_1, z_2, ..., z_k; i) \neq 0$, *then*

$$\left(\frac{W(z, z_1, z_2, ..., z_{k-1}; i)}{W(z_1, z_2, ..., z_k; i)}\right)' = \frac{-W(z_1, z_2, ..., z_{k-1}; i)\, W(z, z_1, ..., z_k; i)}{\rho_{i+k} W^2(z_1, z_2, ..., z_k; i)}.$$

LEMMA 3.2. *If* $W(z_1, z_2, ..., z_{k-1}; i) \neq 0$ *and* $W(z_2, z_3, ..., z_k; i) \neq 0$ *then* $W(z_2, z_3, ..., z_k; i)\, W(z_1, z_2, ..., z_{k-1}, z; i) = W(z_1, z_2, ..., z_{k-1}; i)\, W(z_2, z_3, ..., z_k, z; i) + W(z_2, z_3, ..., z_{k-1}, z; i)\, W(z_1, z_2, ..., z_k; i)$.

LEMMA 3.3. *The set* $\{L_i z_1, L_i z_2, ..., L_i z_n\}$ *forms a Descartes system on an interval I if*

$$W(z_k, z_{k+1}, ..., z_m; i) > 0, \qquad 1 \leqslant k \leqslant m \leqslant n \text{ on } I.$$

THEOREM 3.2. *Let* $b > a$ *be as in Theorem* 3.1. *The set* $\{L_{\gamma-j} u_\gamma(x, b), L_{\gamma-j} u_{\gamma+1}(x, b), ..., L_{\gamma-j} u_{n-1}(x, b)\}$ *(replacing* $u_i(x, b)$ *by* $-u_i(x, b)$ *if necessary) is a Descartes system on* (a, b).

Proof. We use induction. Assume $\{L_{\gamma-j} u_\gamma(x, b), L_{\gamma-j} u_{\gamma+1}(x, b), ..., L_{\gamma-j} u_k(x, b)\}$ forms a Descartes system on (a, b) for $k < n-1$. We need to show $\{L_{\gamma-j} u_\gamma(x, b), L_{\gamma-j} u_{\gamma+1}(x, b), ..., L_{\gamma-j} u_{k+1}(x, b)\}$ also forms a Descartes system there. By the inductive hypothesis and Lemma 3.3, it is enough to prove

$$W(u_s(x, b), u_{s+1}(x, b), ..., u_{k+1}(x, b); \gamma - j) > 0, \qquad \gamma \leqslant s \leqslant k+1 \text{ on } (a, b). \tag{3.2}$$

If $s = \gamma$, this follows from Theorem 3.1. If $s > \gamma$, assume (3.2) holds for smaller values of s. We apply Lemma 3.1 with the following identification:

$$\begin{aligned} z_1 &= u_s \\ z_2 &= u_{s+1} \\ &\vdots \\ z_{m-1} &= u_k \\ z_m &= u_{s-1} \\ z &= u_{k+1}. \end{aligned}$$

Then

$$\left(\frac{W(u_{k+1}, u_s, u_{s+1}, \ldots, u_k; \gamma - j)}{W(u_s, u_{s+1}, \ldots, u_k, u_{s-1}; \gamma - j)}\right)'$$
$$= \frac{-W(u_s, u_{s+1}, \ldots, u_k; \gamma - j)\, W(u_{k+1}, u_s, u_{s+1}, \ldots, u_k, u_{s-1}; \gamma - j)}{\rho_{\gamma - j + k - s + 2}\, W^2(u_s, u_{s+1}, \ldots, u_k, u_{s-1}; \gamma - j)}.$$

Hence

$$\frac{(-1)^{k+1-s}}{(-1)^{k-s+1}}\left(\frac{W(u_s, u_{s+1}, \ldots, u_k, u_{k+1}; \gamma - j)}{W(u_{s-1}, u_s, u_{s+1}, \ldots, u_k; \gamma - j)}\right)'$$
$$= \frac{-W(u_s, u_{s+1}, \ldots, u_k; \gamma - j)\, W(u_{s-1}, u_s, u_{s+1}, \ldots, u_k, u_{k+1}; \gamma - j)}{\rho_{\gamma - j + k - s + 2}\, W^2(u_{s-1}, u_s, u_{s+1}, \ldots, u_k; \gamma - j)}$$
$$\times (-1)^{k-s+2}(-1)^{k-s+1}.$$

Thus, by the inductive hypotheses

$$\frac{W(u_s, u_{s+1}, \ldots, u_k, u_{k+1}; \gamma - j)}{W(u_{s-1}, u_s, u_{s+1}, \ldots, u_k; \gamma - j)}$$

is increasing on (a, b). The denominator is positive by the inductive hypothesis.

Let

$$z_i = \frac{L_{\gamma - j} u_{s-1+i}}{L_{\gamma - j} u_{s-1}}, \qquad i = 1, \ldots, n - s.$$

Then as can be shown by methods as in [1. Lemma 3, p. 87]

$$\frac{W(u_s, u_{s+1}, \ldots, u_k, u_{k+1}; \gamma - j)}{W(u_{s-1}, u_s, u_{s+1}, \ldots, u_k; \gamma - j)}$$
$$= \frac{\begin{vmatrix} z_1 & z_2 & \cdots & z_{k+2-s} \\ \rho_{\gamma-j+1} z_1' & \rho_{\gamma-j+1} z_2' & \cdots & \rho_{\gamma-j+1} z_{\gamma-j+1}' \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{\gamma-j+k-s+1}(\cdots(z_1')\cdots)' & \rho_{\gamma-j+k-s+1}(\cdots(z_2')\cdots)' & \cdots & \rho_{\gamma-j+k-s+1}(\cdots(z_{k+2-s}')\cdots)' \end{vmatrix}}{\begin{vmatrix} 1 & z_1 & \cdots & z_{k+1-s} \\ 0 & \rho_{\gamma-j+1} z_1' & \cdots & \rho_{\gamma-j+1} z_{k+1-s}' \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \rho_{\gamma-j+k-s+1}(\cdots(z_1')\cdots)' & \cdots & \rho_{\gamma-j+k-s+1}(\cdots(z_{k+1-s}')\cdots)' \end{vmatrix}} \tag{3.3}$$

Since $L_{\gamma-j}u_i$ has a zero of multiplicity exactly $i-\gamma+j$ at $x=a$, it follows that z_i has a zero of order exactly i at $x=a$. Thus at $x=a$, where (3.3) in general is indeterminant, the numerator of the expression following the equal in (3.3) is zero while the denominator is nonzero down the diagonal and zero above the diagonal. Thus, it follows that the expression following (3.3) is zero at $x=a$. Since (3.3) is increasing, it is positive on (a, b).

THEOREM 3.3. *Let $b>a$ be as in Theorem 3.1. The set* $\{L_{\gamma-j+1}u_{\gamma+1}(x, b), L_{\gamma-j+1}u_{\gamma+2}(x, b), ..., L_{\gamma-j+1}u_{n-1}(x, b)\}$ where u_i for $i=\gamma+1, ..., n-1$ is the same as in Theorem 3.3 is a Descartes system.

Proof. Applying the proof of Theorem 3.2, $\{L_{\gamma-j+1}u_{\gamma+1}(x, b), L_{\gamma-j+1}u_{\gamma+2}(x, b), ..., L_{\gamma-j+1}u_{n-1}(x, b)\}$, possibly replacing u_i with $-u_i$, is a Descartes system. Since the u_i for $i=\gamma+1, ..., n-1$ of Theorems 3.2 and 3.3 differ by at most a sign and $L_iu_i(a)>0$ for $i=\gamma+1, ..., n-1$, it follows that they are exactly the same.

We now turn to the main results of this section.

THEOREM 3.4. *Suppose $\theta_{\gamma+1}(a, 0)$ fails to exist. Let $\{u_i(x, c)$: $i=0, ..., n-1\}$ be the basis (1.8), with $u_i(x, c)$ positive to the right of a. Let*

$$\hat{u}_\gamma(x, c)=u_\gamma(x, c)-\frac{L_{\gamma+1-j}u_\gamma(c, c)}{L_{\gamma+1-j}u_{\gamma+1}(c, c)}\,u_{\gamma+1}(x, c).$$

Let v_i for $i=0, 1, ..., n-1$ be a basis constructed as in (1.8′) so that for some sequence $\{c_m\}$ diverging to ∞

$$\lim_{m\to\infty} u_i(x, c_m)=v_i(x) \qquad \textit{for} \quad i\neq\gamma$$

$$\lim_{m\to\infty} \hat{u}_\gamma(x, c_m)=v_\gamma(x)$$

with v_i positive just to the right of a. If $z_\gamma(x, b)$ satisfies

$$L_iz_\gamma(a, b)=0, \qquad i=0, 1, ..., \gamma-1, \tag{3.4}$$

$$L_iz_\gamma(b, b)=0, \qquad i=\gamma-j, \gamma-j+1, ..., n-2-j, \tag{3.5}$$

then

$$z_\gamma(x, b)=v_\gamma(x)+\sum_{i=\gamma+1}^{n-1} a_i(b)\,v_i(x),$$

where $a_{\gamma+1}(b)\leqslant 0$.

Proof. We note that the assumption that $\theta_{\gamma+1}(a, 0)$ fails to exist implies that $\theta_{\gamma+1}(a, j)$ fails to exist for all j [3]. Thus $\hat{u}_\gamma(x, c)$ is well defined.

Clearly $z_\gamma(x, b) = v_\gamma(x) + \sum_{i=\gamma+1}^{n-1} a_i(b)\, v_i(x)$. From (3.5)

$$0 = L_t v_\gamma(b) + \sum_{i=\gamma+1}^{n-1} a_i(b)\, L_t v_i(b) \qquad \text{for} \quad t = \gamma - j, ..., n-2-j.$$

Thus, by Cramer's rule

$$-a_{\gamma+1}(b) = \frac{W(v_\gamma(b), v_{\gamma+2}(b), ..., v_{n-1}(b); \gamma - j)}{W(v_{\gamma+1}(b), ..., v_{n-1}(b); \gamma - j)}.$$

Now consider

$$W_1(x, s) = W(\hat{u}_\gamma(x, s), u_{\gamma+2}(x, s), ..., u_{n-1}(x, s); \gamma - j)$$

and

$$W_2(x, s) = W(u_{\gamma+1}(x, s), u_{\gamma+2}(x, s), ..., u_{n-1}(x, s); \gamma - j).$$

Now $W_i(x, s)$ is continuous in both variables and

$$-a_{\gamma+1}(b) = \frac{W(v_\gamma(b), v_{\gamma+2}(b), ..., v_{n-1}(b); \gamma - j)}{W(v_{\gamma+1}(b), ..., v_{n-1}(b); \gamma - j)} = \lim_{n \to \infty} \frac{W_1(b, c_n)}{W_2(b, c_n)}.$$

Let $s = s^*$ be a value of s close enough to a so that no nontrivial solution of (1.1) has more that $n-1$ quasi-derivatives that vanish on $[a, s^*]$. Then $L_{\gamma+2\eta+1-j} u_{\gamma+2\eta+1}(s^*, s^*) \neq 0$. Let

$$\hat{u}_{\gamma+2\eta}(x, s^*) = u_{\gamma+2\eta}(x, s^*) - \frac{L_{\gamma+2\eta+1-j} u_{\gamma+2\eta}(s^*, s^*)}{L_{\gamma+2\eta+1-j} u_{\gamma+2\eta+1}(s^*, s^*)}\, u_{\gamma+2\eta+1}(x, s^*) \tag{3.6}$$

and

$$\hat{u}_{\gamma+2\eta+1}(x, s^*) = u_{\gamma+2\eta+1}(x, s^*) \qquad \text{for} \quad \gamma + 2\eta + 1 \leqslant n - 1.$$

Let

$$\hat{u}_{n-1}(x, s^*) = u_{n-1}(x, s^*).$$

Note that for $x \in (a, s^*)$ and close enough to a, $\hat{u}_{\gamma+2\eta}(x, s^*) > 0$ since $u_{\gamma+2\eta}(x, s^*) > 0$. It now follows from Theorems 3.2 and 3.3 that

$$\{L_{\gamma-j} \hat{u}_\gamma(x, s^*), L_{\gamma-j} \hat{u}_{\gamma+1}(x, s^*), ..., L_{\gamma-j} \hat{u}_{n-1}(x, s^*)\} \tag{3.7}$$

and

$$\{L_{\gamma-j+1}\hat{u}_{\gamma+1}(x, s^*), L_{\gamma-j+1}\hat{u}_{\gamma+2}(x, s^*), ..., L_{\gamma-j+1}\hat{u}_{n-1}(x, s^*)\} \quad (3.8)$$

are Descartes systems on (a, s^*).

By (3.6) and (3.7), for $a < x < s^*$

$$0 < W(\hat{u}_{\gamma+1}(x, s^*), \hat{u}_{\gamma+2}(x, s^*), ..., \hat{u}_{n-1}(x, s^*); \gamma - j) = W_2(x, s^*).$$

Now $W_2(x, s) \neq 0$ for $x \in (a, s]$, $a < s < \infty$. Otherwise there is $x^* \in (a, s]$ so that

$$L_i u(a) = 0, \qquad i = 0, 1, ..., \gamma$$

$$L_i u(x^*) = 0, \qquad \gamma - j, \gamma - j + 1, ..., n - 2 - j,$$

contrary to the hypothesis. Thus $W_2(x, s) > 0$ for $a < x \leqslant s < \infty$.

Now $W(\hat{u}_\gamma(x^*, s), u_{\gamma+1}(x^*, s), ..., u_{n-1}(x^*, s); \gamma - j) \neq 0$ for $a < x^* < s < \infty$. Otherwise there is a solution to the boundary value problem

$$L_i\, y(a) = 0, \qquad i = 0, 1, ..., \gamma - 1$$

$$L_i\, y(x^*) = 0, \qquad i = \gamma - j, \gamma - j + 1, ..., n - 1 - j,$$

which is not possible because of the sign condition on $p(x)$. For the same reasons $W(u_{\gamma+2}(x^*, s), ..., u_{n-1}(x^*, s); \gamma - j) \neq 0$ for $a < x^* \leqslant s$. Thus by (3.7) and (3.6) it follows that $W(\hat{u}_\gamma(x, s), u_{\gamma+1}(x, s), ..., u_{n-1}(x, s); \gamma - j) > 0$ and $W(u_{\gamma+2}(x, s), ..., u_{n-1}(x, s); \gamma - j) > 0$ for $a < x \leqslant s < \infty$.

For any fixed s, applying Lemma 3.1 yields

$$\left(\frac{W_1(x, s)}{W_2(x, s)}\right)' = (-1)\frac{\begin{array}{c}(W(u_{\gamma+2}(x, s), ..., u_{n-1}(x, s); \gamma - j)\; W(\hat{u}_\gamma(x, s),\\ u_{\gamma+1}(x, s), ..., u_{n-1}(x, s); \gamma - j))\end{array}}{\rho_{n-j-1}(x)\; W_2^2(x, s)}.$$

Hence, $W_1(x, s)/W_2(x, s)$ is decreasing in x on (a, s).

Now

$$W_1(s, s) = L_{\gamma-j}\hat{u}_\gamma(s, s)\; W(u_{\gamma+2}(s, s), ..., u_{n-1}(s, s); \gamma - j + 1).$$

Again, the sign condition on $p(x)$ prevents $W(u_{\gamma+2}(x, s), ..., u_{n-1}(x, s); \gamma - j + 1)$ from being zero for $a < x \leqslant s < \infty$. But again by (3.8) $W(u_{\gamma+2}(x, s), ..., u_{n-1}(x, s); \gamma - j + 1) > 0$ for s near a. Thus $W(u_{\gamma+2}(x, s), ..., u_{n-1}(x, s); \gamma - j + 1) > 0$ for all $s > a$.

There is an $s > a$ so that $L_{\gamma-j}\hat{u}_\gamma(x, s) \neq 0$ for $a < x < s$. If $j = 0$ then the existence of such a zero implies all quasi-derivatives $L_0\hat{u}_\gamma(x, s)$, $L_1\hat{u}_\gamma(x, s)$, ..., $L_{n-1}\hat{u}_\gamma(x, s)$ vanish on $[a, s]$ which is not possible. If $j > 0$ then applying Rolle's theorem leads to the same contradiction.

If for some s^*, $L_{\gamma-j}\hat{u}_\gamma(x^*, s^*)=0$ for some $a<x^*<s^*$, x^* must be a simple zero and thus continuous as a function of s. Thus as s moves to the left toward a it must exit at either a or s. If the zero approaches a then we will have a solution satisfying the boundary conditions

$$\begin{aligned} L_i y(a)&=0, \qquad i=0, ..., \gamma \\ L_i y(s)&=0, \qquad i=\gamma+1-j, ..., n-1-j, \end{aligned}$$

contrary to assumptions. If the zero x^* approaches s, we have a solution of

$$\begin{aligned} L_i y(a)&=0, \qquad i=0, ..., \gamma-1 \\ L_i y(s)&=0, \qquad i=\gamma-j, ..., n-1-j, \end{aligned}$$

which is not possible because of the sign condition on $p(x)$. It follows that $L_{\gamma-j}\hat{u}_\gamma(x, s)\neq 0$ for $a<x<s<\infty$.

By (3.7) $L_{\gamma-j}\hat{u}_\gamma(x, s)>0$ for s near a and $a<x\leqslant s$. Thus, $L_{\gamma-j}\hat{u}_\gamma(x, s)>0$ for all $a<x\leqslant s<\infty$. Hence $W_1(s, s)>0$. Since $W_2(s, s)>0$ and $W_1(x, s)/W_2(x, s)$ is decreasing as a function of x on (a, s), it follows that $W_1(x, s)/W_2(x, s)>0$ for $a<x<s$. Thus

$$\begin{aligned} -a_{\gamma+1}(b)&=\frac{W(v_\gamma(b), v_{\gamma+2}(b), ..., v_{n-1}(b); \gamma-j)}{W(v_{\gamma+1}(b), ..., v_{n-1}(b); \gamma-j)} \\ &=\lim_{n\to\infty}\frac{W_1(b, c_n)}{W_2(b, c_n)}\geqslant 0. \end{aligned}$$

THEOREM 3.5. *Suppose $\theta_{\gamma+1}(a, 0)$ fails to exist and let $a_{\gamma+1}$ be as in Theorem 3.4. Then*

$$\lim_{b\to\infty} a_{\gamma+1}(b)=0.$$

Proof. We will prove the theorem with the added hypothesis that

$$\lim_{x\to\infty}\frac{L_i v_\gamma(x)}{L_i v_{\gamma+1}(x)}=0 \qquad \text{for} \quad i=0, ..., n-1, \tag{3.9}$$

where $\{v_0, v_1, ..., v_{n-1}\}$ are as in Theorem 3.4. We need to show

$$\lim_{x\to\infty}\frac{W(v_\gamma(x), v_{\gamma+2}(x), ..., v_{n-1}(x); \gamma-j)}{W(v_{\gamma+1}(x), ..., v_{n-1}(x); \gamma-j)}=0.$$

We let

$$W_1(x)=W(v_\gamma(x), v_{\gamma+2}(x), ..., v_{n-1}(x); \gamma-j)$$

and

$$W_2(x) = W(v_{\gamma+1}(x), \ldots, v_{n-1}(x); \gamma - j).$$

From the proof of Theorem 3.4, $W_1(x)/W_2(x)$ is a nonnegative nonincreasing function. Suppose

$$\lim_{x \to \infty} \frac{W_1(x)}{W_2(x)} = \Lambda > 0.$$

Let $\{c_m\} \to \infty$ and $\Lambda_m = W_1(c_m)/W_2(c_m)$. Then $\lim_{m \to \infty} \Lambda_m = \Lambda$. Let $D_m(x) = W_1(x) - \Lambda_m W_2(x)$. Then $D_m(c_m) = 0$. Hence there is a solution z_m of (1.1) in $\operatorname{span}\{v_\gamma - \Lambda_m v_{\gamma+1}, v_{\gamma+2}, \ldots, v_{n-1}\}$ so that

$$L_i z_m(c_m) = 0, \qquad i = \gamma - j, \gamma - j + 1, \ldots, n - 2 - j. \tag{3.10}$$

Without loss of generality, assume $z_m \to z$ in $\operatorname{span}\{v_\gamma - \Lambda v_{\gamma+1}, v_{\gamma+2}, \ldots, v_{n-1}\}$ where z is nontrivial.

Let $x < c_m$. Then $n \geqslant S(z_m, x^+) + S(z_m, c_m^-) \geqslant S(z_m, x^+) + n - \gamma - 1$. Thus

$$S(z_m, x^+) \leqslant \gamma + 1. \tag{3.11}$$

It follows by [2] that

$$S(z, x^+) \leqslant \gamma + 1. \tag{3.12}$$

Suppose

$$z = b_{\gamma+1}(v_\gamma - \Lambda v_{\gamma+1}) + b_{\gamma+2} v_{\gamma+2} + \cdots + b_{n-1} v_{n-1}. \tag{3.13}$$

Note that $b_{\gamma+1} \neq 0$. Otherwise z would be in $\operatorname{span}\{v_{\gamma+2}, \ldots, v_{n-1}\}$ and thus $S(z, x^+) \geqslant \gamma + 3$ contrary to (3.12).

Now $L_{\gamma-j} z_m(x) \neq 0$ for $a < x < c_m$. Otherwise there is an x_1, $a < x_1 < c_m \equiv s$, where $L_{\gamma-j} z_m(x_1) = 0$. First, such a zero would have to be simple. Otherwise, $n \geqslant S(z_m, a^+) + \langle n(x_1) \rangle + S(z_m, c_m^-) \geqslant (\gamma+1) + 2 + [n - (\gamma+1)] = n + 2$. On the other hand, simple zeros of $L_{\gamma-j} z_m$ are continuous functions of s. Letting s move toward a from the right until x_1 exits (a, s) or until $s = a$, we get (after normalizing) a nontrivial limit function u in $\operatorname{span}\{v_\gamma, v_{\gamma+1}, \ldots, v_{n-1}\}$ with either

$$L_i u(a) = 0, \qquad \text{for} \quad i = 0, 1, \ldots, \gamma, \tag{3.14}$$

$$L_i u(s) = 0, \qquad \text{for} \quad i = \gamma - j, \gamma - j + 1, \ldots, n - 1 - j, \tag{3.15}$$

or

$$L_i u(a) = 0, \qquad \text{for} \quad i = 0, 1, \ldots, n - 1. \tag{3.16}$$

In case (3.14), u would be in $\operatorname{span}\{v_{\gamma+1}, v_{\gamma+2}, \ldots, v_{n-1}\}$. But since $W_1(s)/W_2(s) \neq 0$, $u \in \operatorname{span}\{v_{\gamma+2}, \ldots, v_{n-1}\}$. Thus, $S(u, a^+) + S(u, s^-) \geqslant (\gamma+3) + (n-\gamma-1) = n+2$.

In the case (3.15), $n \geqslant S(u, a^+) + S(u, s^-) \geqslant (\gamma+1) + (n-\gamma) = n+1$.

The last case is impossible for nontrivial solutions.

Thus, assume without loss of generality that $L_{\gamma-j} z_m(x) > 0$ for $a < x < c_m$. It follows that $L_{\gamma-j} z(x) \geqslant 0$ for $a \leqslant x$. By (3.12) and (3.13)

$$S(z, x^+) = \gamma + 1 \qquad \text{for} \quad x > a. \tag{3.17}$$

Hence $L_{\gamma-j} z(x) > 0$ since $L_{\gamma-j} z$ can have no double zeros.

If $\sum_{i=2}^{n-1-\gamma} b_{\gamma+i} v_{\gamma+i}$ is nonoscillatory, then known dominance properties give $S(z, x^+) \geqslant \gamma + 3$ contrary to (3.17). If $\sum_{i=2}^{n-1-\gamma} b_{\gamma+i} v_{\gamma+i}$ is oscillatory, then since

$$\lim_{x \to \infty} \frac{L_{\gamma-j} v_\gamma(x)}{L_{\gamma-j} v_{\gamma+1}(x)} = 0,$$

for large zeros of $L_{\gamma-j}(\sum_{i=2}^{n-1-\gamma} b_{\gamma+i} v_{\gamma+i})$, sgn $L_{\gamma-j} z$ must be opposite that of $L_{\gamma-j} b_{\gamma+1} v_{\gamma+1}$; however, to the right of a, sgn $L_{\gamma-j} z$ must be the same as that of $L_{\gamma-j} b_{\gamma+1} v_\gamma$. In that case $L_{\gamma-j} z$ has a zero, which is not possible.

Corollary. *With $a_i(x)$ as in Theorem 3.4,*

$$\lim_{x \to \infty} a_i(x) = 0$$

for $i = \gamma+1, \gamma+2, \ldots, n-1$.

Proof. Let v_i and $z_\gamma(x, b)$ be as in Theorem 3.4. Assuming $\lim_{x \to \infty} (L_i v_\gamma(x)/L_i v_{\gamma+1}(x)) = 0$, Theorem 3.5 shows that (letting $b \to \infty$)

$$z_\gamma(x) = v_\gamma(x) + \sum_{i=\gamma+2}^{n-1} a_i v_i(x) \equiv v_\gamma(x) + z(x).$$

Now $S(v_\gamma, x^+) = \gamma + 1$ while $S(z, x^+) \geqslant \gamma + 3$ and $S(z_\gamma, x^+) = \gamma + 1$. Thus

$$L_i(v_\gamma(x) + z(x)) > 0, \qquad i = 0, 1, \ldots, \gamma$$
$$(-1)^{i-\gamma} L_i(v_\gamma(x) + z(x)) > 0, \qquad i = \gamma+1, \ldots, n-1.$$

for all large x. Hence for every $\varepsilon > 0$, eventually

$$L_i(\varepsilon v_{\gamma+1}(x) + z(x)) > L_i(v_\gamma(x) + z(x)) > 0, \qquad \text{for} \quad i = 0, 1, \ldots, \gamma$$

and

$$(-1)^{i-\gamma} L_i(\varepsilon v_{\gamma+1}(x)+z(x)) > (-1)^{i-\gamma} L_i(v_\gamma(x)+z(x))$$
$$>0, \qquad \text{for} \quad i=\gamma+1, \ldots, n-1.$$

But that says that $S(\varepsilon v_{\gamma+1}+z, x^+)=\gamma+1$ for every $\varepsilon>0$, which is not possible by [2].

THEOREM 3.6. *The assumption that*

$$\lim_{x\to\infty} \frac{L_i v_\gamma(x)}{L_i v_{\gamma+1}(x)}=0 \tag{3.18}$$

is not needed in Theorem 3.5 *and its corollary.*

Proof. To prove the theorem, we need to show a relationship between basis elements for various values of j. Let $\{v_{i,j}(x): i=0, \ldots, n-1\}$ be a basis for the solution space of (1.1) as in Theorem 3.4; i.e.,

$$v_{i,j}(x)=\lim_{m\to\infty} v_{i,j}(x, c_m(j)),$$

where $\{c_m(j)\}$ diverges to infinity with m and $v_{i,j}(x, c_m(j))$ satisfies (1.9)–(1.12) with $b=c_m(j)$ and $i\neq\gamma$. However $v_{\gamma,j}(x, c_m(j))$ satisfies

$$L_i v_{\gamma,j}(x, c_m(j))=0 \qquad \text{for} \quad i=0, \ldots, \gamma-1 \tag{3.19}$$

and

$$L_i v_{\gamma,j}(c_m(j), c_m(j))=0 \qquad \text{for} \quad i=\gamma+1-j, \ldots, n-1-j. \tag{3.20}$$

If we assume

$$\lim_{x\to\infty} \frac{L_i v_{\gamma,j}(x)}{L_i v_{\gamma+1,j}(x)}=0 \qquad \text{for} \quad i=0, \ldots, n-1, \tag{3.21}$$

then by the corollary to Theorem 3.5 and Theorem 3.4

$$v_{\gamma,j}(x)=v_{\gamma,j+1}(x).$$

Now

$$v_{\gamma+1,j+1}(x)=v_{\gamma+1,j}(x)+z, \tag{3.22}$$

where

$$z=\sum_{i=\gamma+2}^{n-1} b_i v_{i,j}.$$

If z is nonoscillatory, since $S(z, x^+) \geqslant \gamma + 3$, by [2]

$$\lim_{x \to \infty} \frac{v_{\gamma+1, j+1}(x)}{z(x)} = \lim_{x \to \infty} \frac{v_{\gamma+1, j}(x)}{z(x)} = 0.$$

Thus by (3.22) $z \equiv 0$. If z is oscillatory, let $\{x_{i,m}\}$ be a sequence diverging to ∞ with m so that

$$L_i z(x_{i,m}) = 0 \qquad \text{for all} \quad m.$$

Then

$$\lim_{m \to \infty} \frac{L_i v_{\gamma, j+1}(x_{i,m})}{L_i v_{\gamma+1, j+1}(x_{i,m})} = \lim_{m \to \infty} \frac{L_i v_{\gamma, j}(x_{i,m})}{L_i v_{\gamma+1, j}(x_{i,m})} = 0.$$

But by Theorem 1.2, $L_i v_{\gamma, j+1}(x)/L_i v_{\gamma+1, j+1}(x)$ is monotone. Thus

$$\lim_{x \to \infty} \frac{L_i v_{\gamma, j+1}(x)}{L_i v_{\gamma+1, j+1}(x)} = 0, \qquad i = 0, 1, ..., n-1.$$

In the case $j = 0$, (3.21) is known to hold [2]. Thus the result follows by induction.

Acknowledgment

The authors wish to acknowledge the referee's many helpful comments.

References

1. W. A. Coppel, "Disconjugacy," Lecture Notes in Math., Vol. 220, Springer-Verlag, Berlin/New York, 1971.
2. U. Elias, A classification of the solutions of a differential equation according to their asymptotic behavior, *Proc. Roy. Soc. Edinburgh Sect. A* **83** (1979), 25–38.
3. U. Elias, Oscillatory solutions and extremal points for linear differential equations, *Arch. Rational Mech. Anal.* **71** (1979), 177–198.
4. U. Elias, Minors of the wronskian of the differential equation $L_n y + p(x) y = 0$, *Proc. Roy. Soc. Edinburgh Sect. A* **106** (1987), 341–359.
5. U. Elias, Minors of the wronskian of the differential equation $L_n y + p(x) y = 0$ II; Dominance of solutions, *Proc. Roy. Soc. Edinburgh Sect. A*, in press.
6. G. Etgen, G. Jones, and W. E. Taylor, Jr., On the factorizations of ordinary linear differential equations, *Trans. Amer. Math. Soc.* **297** (1986), 717–728.
7. W. F. Trench, Canonical forms and principal systems for general disconjugate equations, *Trans. Amer. Math. Soc.* **189** (1974), 319–327.